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# Strongly Simply Connected Algebras

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## INTRODUCTION

In the representation theory of finite dimensional algebras over an algebraically closed field, the simply connected representation-finite algebras introduced by Bongartz and Gabriel [6] have played an important rôle (see, for instance, [5], [7]). The reason for their importance is that, for a representation-finite algebra $A$, the indecomposable $A$-modules can be lifted to indecomposable modules over a simply connected algebra $\tilde{A}$ (contained inside a certain Galois covering of the standard form of $A$; see [7]). Thus, covering techniques allow us to reduce many problems of the study of representation-finite algebras to problems about simply connected representation-finite algebras. Little is known about covering techniques or simply connected algebras in the representation-infinite case. One class, however, of simply connected algebras has attracted much interest lately; this is the class of strongly simply connected algebras, introduced by Skowroński in [14]. The representation theory of strongly simply connected algebras seems to be relatively accessible, and some progress has been made in understanding it in the tame case (see, for instance, [13], [15]).

The purpose of this paper is to provide characterizations and construction techniques for strongly simply connected algebras. Since we are



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motivated by the study of coverings, we start by considering locally bounded $k$-categories [6] and give an alternative definition for the strong simple connectedness of a locally bounded category (Section 1.3), which we believe is easier to handle than the one in [14]. We then show the equivalence of these two definitions, and, while doing so, we obtain a handy criterion allowing us to verify whether a locally bounded category is strongly simply connected (Theorem 1.3). We next consider the case of Schurian locally bounded categories. We recall that the Schurian strongly simply connected algebras were already studied in [9], under the name of completely separating algebras. Here, we prove that a connected triangular locally bounded category is Schurian and strongly simply connected if and only if it has a presentation (called normed presentation; see [4]) such that all cycles are commutative (Theorem 2.4). We deduce a new necessary and sufficient condition for a representation-finite algebra to be simply connected (Corollary 2.5). We then turn our attention to the construction of strongly simply connected algebras. Since such an algebra is triangular, it can be constructed by repeated one-point extensions or coextensions. We define in Definition 3.3 a notion of a completely coseparated module, and the dual notion of a completely separated module. Our main theorem (Theorem 3.4) states that an algebra is strongly simply connected if and only if it is the one-point extension (or coextension) of a strongly simply connected algebra by a completely coseparated module (or a completely separated module, respectively). We end the paper with an inductive construction of the Schurian strongly simply connected algebras with a prescribed number of isomorphism classes of simple modules (Theorem 4.4).

## 1. STRONGLY SIMPLY CONNECTED LOCALLY BOUNDED CATEGORIES

### 1.1. *Locally bounded categories*

Throughout this paper, $k$ denotes a fixed algebraically closed field. We recall that a $k$-category $A$ is a category where, for each pair of objects $x, y$ of $A$, the set of morphisms $A(x, y)$ from $x$ to $y$ has a $k$-vector space structure such that the composition of morphisms is $k$-bilinear. Let $A_0$ denote the class of objects of $A$. A $k$-category $A$ is called *locally bounded* [6] if (a) for each $x \in A_0$, the endomorphism algebra $A(x, x)$ is local; (b) distinct objects are not isomorphic; and (c) for each $x \in A_0$, we have $\sum_{y \in A_0} \dim_k A(x, y) < \infty$ and $\sum_{y \in A_0} \dim_k A(y, x) < \infty$.

Locally bounded categories are realized by locally finite quivers: if $A$ is a locally bounded category, there exist a locally finite quiver $Q_A$ and an admissible ideal $I$ of the path category $kQ_A$ of $Q_A$ such that we have an isomorphism $A \cong kQ_A/I$, called a *presentation* of $A$. The pair $(Q_A, I)$ is then called a *bound quiver*. We recall that a *quiver* $Q$ is defined by its set of points $Q_0$, its set of arrows $Q_1$, and two mappings $Q_1 \to Q_0$ associating with each arrow its source and its target, respectively. If the quiver $Q$ is finite and connected, a bound quiver category $kQ/I$ can equivalently be viewed as a finite-dimensional $k$-algebra, which is, moreover, basic and connected. Conversely, any finite-dimensional basic and connected $k$-algebra occurs in this way [10].

Let $A$ be a locally bounded category. A full subcategory $B$ of $A$ is called *convex* if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in (the quiver of) $A$ with $x_0, x_t \in B_0$, we have $x_i \in B_0$ for all $1 \leq i < t$. The category $A$ is called *triangular* if its quiver $Q_A$ contains no oriented cycle.

By an $A$-module is meant a finitely generated right $A$-module. We denote here by mod $A$ their category. It is well known that if $A = kQ/I$, then mod $A$ is equivalent to the category of all bound (finite-dimensional) representations of $(Q, I)$ (see [6], [10]). For each $x \in Q_0$, we denote by $P(x)$ the corresponding indecomposable projective $A$-module.

## 1.2. *The fundamental group*

Let $(Q, I)$ be a connected locally finite bound quiver. A *relation* from a point $x$ to a point $y$ is an element $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ such that, for each $1 \leq i \leq m$, $\lambda_i$ is a nonzero scalar and $w_i$ is a path of length at least two from $x$ to $y$. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is called *minimal* if $m \geq 2$ and, for any proper nonempty subset $J \subseteq \{1, 2 \dots m\}$, we have $\sum_{j \in J}^{i=1} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by $\alpha^{-1}$ its formal inverse. A *walk* in $Q$ from $x$ to $y$ is a formal composition $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2} \cdots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$, $\varepsilon_i = \pm 1$ for all $1 \leq i \leq t$) starting at $x$ and ending at $y$. We denote by $e_x$ the trivial path at $x$. A walk in $Q$ is called *reduced* if it contains no subwalk of one of the forms $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ with $\alpha \in Q_1$.

Let $\sim$ be the least equivalence relation in the set of all walks in $Q$ such that

(a) If $\alpha\colon x \to y$ is an arrow, then $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.

(b) If $\sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all $1 \leq i, j \leq m$.

(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions are defined.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes of all walks starting and ending at $x$ has a group structure with operation induced from the composition of walks. Since, clearly, the group $\pi_1(Q, I, x)$

does not depend on the choice of $x$, we denote it by $\pi_1(Q, I)$ and call it the *fundamental group* of $(Q, I)$ (see [11], [12]).

### 1.3. *Strong simple connectedness*

Let $Q$ be a locally finite quiver. A full subquiver $Q'$ of $Q$ is called *convex* if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in $Q$, with $x_0, x_t \in Q'_0$, we have $x_i \in Q'_0$ for all $1 \leq i < t$. A bound quiver $(Q', I')$ is a full bound subquiver of a bound quiver $(Q, I)$ if $Q'$ is a full subquiver of $Q$, and $I' = I \cap kQ'$. We are now ready to define our object of study.

DEFINITION 1.1. A connected triangular locally bounded $k$-category $A$ is called *strongly simply connected* if there exists a presentation $A \cong kQ_A/I_A$ of $A$ such that, for any connected full convex bound subquiver $(Q,I)$ of $(Q_A, I_A)$, we have $\pi_1(Q, I) = 1$.

Thus, if $B$ is a full convex subcategory of $A$, and we denote by $Q_B$ the full subquiver of $Q_A$ generated by the set of points in $Q_A$ that correspond to objects in $B$, and by $I_B$ the ideal $I_B = kQ_B \cap I_A$, we have $\pi_1(Q_B, I_B) = 1$.

For example, a hereditary (or a monomial) locally bounded category is strongly simply connected if and only if its quiver is a tree.

We recall that Skowroński has given in [14] another definition of a strongly simply connected finite-dimensional algebra: a triangular algebra $A$ is called *simply connected* if, for any presentation $A \cong kQ_A/I_A$, we have $\pi_1(Q_A, I_A) = 1$ (see [3]); it is called *strongly simply connected* if every connected full convex subcategory of $A$ is simply connected. Our first task is thus to show the equivalence of these two definitions.

For our first lemma, we need the following definition, due to Bautista et al. [5]. Let $A$ be a triangular locally bounded $k$-category (not necessarily connected). An $A$-module $M$ is called *separated* if, for each connected component $C$ of $A$, the restriction $M|_C$ of $M$ to $C$ is zero or is indecomposable. This can be expressed in terms of supports: the *support* of an $A$-module $M$ is the full subcategory Supp $M$ of $A$ generated by all $x \in A_0$ such that $M_x \neq 0$. Thus, an $A$-module $M$ is separated if and only if the supports of the distinct indecomposable summands of $M$ lie in distinct connected components of $A$. For each $x \in Z_0$, let $A^x$ denote the full subcategory of $A$ generated by the nonpredecessors of $x$ in $Q_A$. The object $x$ is called *separating* if the restriction to $A^x$ of rad $P(x)_A$ is separated as an $A^x$-module. We say that $A$ satisfies the *separation condition* if each $x \in A_0$ is a separating object. One defines dually coseparating objects and the coseparation condition.

We also need the following definition. Let $Q$ be a locally finite quiver without oriented cycles. A *contour* $(p, q)$ in $Q$ from $x$ to $y$ is a pair of paths $p, q$ of positive length having the same source $x$ and the same target $y$.

LEMMA 1.2. *Let $A$ be a strongly simply connected locally bounded k-category. Then any connected full convex subcategory of $A$ satisfies the separation condition.*

*Proof.* Let $A \cong kQ_A/I_A$ be a presentation of $A$ such that the fundamental group of any connected full convex bound subquiver of $(Q_A, I_A)$ is trivial. To establish the lemma, it suffices to show that $A$ itself satisfies the separation condition. Suppose, on the contrary, that there exists an object $x \in A_0$ that is not separating. Let $R(x) = \operatorname{rad} P(x)_A$. The $k$-vector space $R(x)$ has as its basis the residue classes modulo $I$ of the paths in $Q_A$ of positive length and source $x$. Let $B$ be a connected component of $A^x$ such that $R(x)|_B$ is decomposable. Assume $R(x)|_B = R_1 \oplus R_2$, with $R_1, R_2$ nonzero. Let $(Q, I)$ be the full bound subquiver of $(Q_A, I_A)$ generated by $B$ and $x$. Then $(Q, I)$ is clearly connected and convex in $(Q_A, I_A)$. Denote by $K$ the Kronecker quiver,

$$b \circ \underset{\delta}{\overset{\gamma}{\leftleftarrows}} \circ a.$$

We complete the proof by constructing a group epimorphism from $\pi_1(Q, I)$ onto $\pi_1(K)$, and this is a contradiction, because $\pi_1(Q, I) = 1$ by hypothesis, while clearly $\pi_1(K) \cong \mathbb{Z}$.

We define a surjective map $\varphi$ from the set of walks in $Q$ onto the set of walks in $K$ as follows. We set $\varphi(e_x) = e_a$ and, for all $y \in Q_0$ such that $y \neq x$, we set $\varphi(e_y) = e_b$. For an arrow $\alpha\colon y \to z$ in $Q$, we let $\varphi(\alpha) = e_b$ if $y \neq x$ and, in the case where $y = x$, we define $\varphi(\alpha) = \gamma$ if $z$ belongs to $\operatorname{Supp} R_1$, and $\varphi(\alpha) = \delta$ if $z$ belongs to $\operatorname{Supp} R_2$. The map $\varphi$ is well defined, since $R_1 \cap R_2 = 0$. Define $\varphi(\alpha^{-1}) = \varphi(\alpha)^{-1}$. For an arbitrary walk $w = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2} \dots \alpha_t^{\varepsilon_t}$ in $Q$ from $y$ to $z$, say, with $\varepsilon_i = \pm 1$, $1 \leq i \leq t$, it is easily shown that

$$\varphi(w) = \varphi(\alpha_1)^{\varepsilon_1}\varphi(\alpha_2)^{\varepsilon_2} \cdots \varphi(\alpha_t)^{\varepsilon_t}$$

is a walk in $K$ from the point corresponding to $\varphi(e_y)$ to the point corresponding to $\varphi(e_z)$.

Now let $(p_1, p_2)$ be a contour in $Q$ from $y$ to $z$ such that there exists a minimal relation $\sum_{i=1}^m \lambda_i p_i$. Write $p_1 = \alpha_1 q_1$, $p_2 = \alpha_2 q_2$ with $\alpha_1, \alpha_2 \in Q_1$. If $y \neq x$, then $\varphi(p_1) = \varphi(p_2) = e_b$, since $x$ is a source in $Q$. Assume that $y = x$. Then $\varphi(p_1) = \varphi(\alpha_1)$ and $\varphi(p_2) = \varphi(\alpha_2)$. If the target $y_1$ of $\alpha_1$ lies in $\operatorname{Supp} R_1$, then $q_1$ lies entirely in $\operatorname{Supp} R_1$. Hence $q_2$ also lies entirely in $\operatorname{Supp} R_1$, because $R_1 \cap R_2 = 0$. This implies that the target of $\alpha_2$ lies in

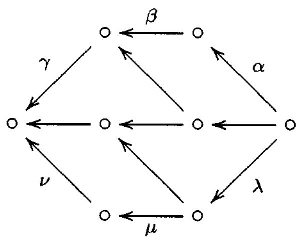
Supp $R_1$. Therefore, $\varphi(p_1) = \varphi(\alpha_1) = \varphi(\alpha_2) = \varphi(p_2) = \gamma$. Similarly, if $y_1$ lies in Supp $R_2$, we have $\varphi(p_1) = \varphi(\alpha_1) = \varphi(\alpha_2) = \varphi(p_2) = \delta$. This shows that $\varphi$ is compatible with the equivalence relation defined on $(Q, I)$ and thus induces as required a group epimorphism $\pi_1(Q, I) \to \pi_1(K)$. ∎

1.5. *Definitions and notations*

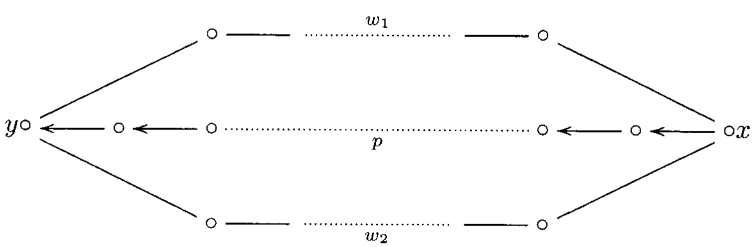
We need a few definitions and notations. Let $Q$ be a locally finite quiver without oriented cycles. By cycle, we mean an unoriented simple cycle, that is, a subquiver $C$ of $Q$ is a *cycle* if each point in $C$ is an endpoint of exactly two arrows in $C$ and there exists an enumeration $\{x_0, x_1, \dots, x_{n-1}, x_n = x_0\}$ of the points of $C$ such that there exists an edge between $x_{i-1}$ and $x_i$ on $C$, for all $1 \le i \le n$.

A contour $(p, q)$ in $Q$ from $x$ to $y$ is called *interlaced* if the paths $p$ and $q$ have a common point other than $x$ and $y$. Thus, a contour is a cycle if and only if it is not interlaced. A contour $(p, q)$ is called *reducible* if there exist paths $p = p_0, p_1, \dots, p_m = q$ in $Q$ from $x$ to $y$ such that, for each $1 \le i \le m$, the contour $(p_{i-1}, p_i)$ is interlaced. In this case, we say that $p$ is *reducible* to $q$. Otherwise, it is called *irreducible*.

In the following example, the contour $(\alpha\beta\gamma, \lambda\mu\nu)$ is reducible:

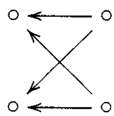



Let $C$ be a cycle that is not a contour. Denote by $\sigma(C)$ the number of sources of $C$ (which actually equals the number of sinks of $C$, by our definition of cycle). Thus $\sigma(C) > 1$. The cycle $C$ is said to be *reducible* if there exist two points $x, y$ in $C$, and a path $p : x \to \cdots \to y$ in $Q$ as follows:



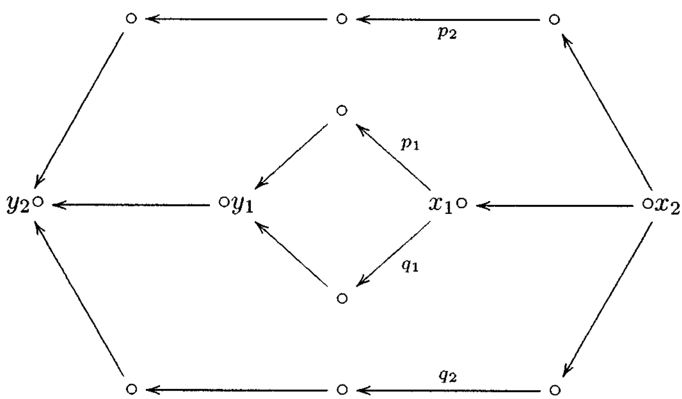
where the cycle $C$ consists of the walks $w_1$ and $w_2$, such that both $w_1 p^{-1}$ and $w_2 p^{-1}$ are cycles and $\sigma(w_1 p^{-1}) < \sigma(C), \sigma(w_2 p^{-1}) < \sigma(C)$. We then say that a path such as $p$ *reduces the cycle* $C$. A cycle $C$ is said to be *irreducible* if it is either an irreducible contour, or it is not a contour, but it is not reducible in the above sense.

A typical example of an irreducible cycle that is not an irreducible contour is as follows:



We also define a partial order on the contours in $Q$ as follows. Let $(p_1, q_1)$ and $(p_2, q_2)$ be two contours from $x_1$ to $y_1$ and $x_2$ to $y_2$, respectively. Then $(p_1, q_1) \leq (p_2, q_2)$ if either $(p_1, q_1) = (p_2, q_2)$ or $(x_1, y_1) \neq (x_2, y_2)$, and then $x_1$ is a successor of $x_2$, and $y_1$ is a predecessor of $y_2$.

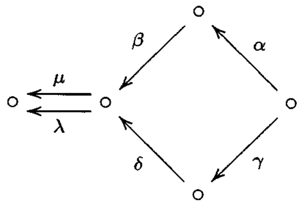
In the following example, we have $(p_1, q_1) \leq (p_2, q_2)$:




The above definitions are purely quiver-theoretical. We also need a notion of contractibility of contours. Let $Q$ be, as before, a locally finite quiver without oriented cycles, and $I$ be an admissible ideal of $kQ$. Two paths $p, q$ from $x$ to $y$ in $Q$ are called *naturally homotopic* in $(Q, I)$ if there exists a sequence of paths $p = p_0, p_1, \dots, p_m = q$ in $Q$ such that, for each

$0 \leq i < m$, $p_i$ and $p_{i+1}$ have subpaths $q_i$ and $q_{i+1}$, respectively, which are involved in the same minimal relation in $(Q, I)$. A contour $(p, q)$ is called *naturally contractible* if the paths $p, q$ are naturally homotopic in $(Q, I)$.

The following example illustrates this definition. Let $Q$ be the quiver




and $I$ be the ideal generated by $\alpha\beta - \gamma\delta$, $\alpha\beta\lambda - \alpha\beta\mu$. The paths $\alpha\beta, \gamma\delta$ are naturally homotopic in $(Q, I)$, and thus the contour $(\alpha\beta, \gamma\delta)$ is naturally contractible. On the other hand, the paths $\lambda, \mu$ are homotopic in $(Q, I)$, but not naturally homotopic, hence the contour $(\lambda, \mu)$ is not naturally contractible.

We are now able to state (and prove) the main result of this section, which asserts that our definition of strong simple connectedness is equivalent to that in [14], and gives a handy criterion allowing us to verify whether a locally bounded category is strongly simply connected.

THEOREM 1.3. *Let $A$ be a connected triangular locally bounded $k$-category. The following conditions are equivalent*:

(a) *$A$ is strongly simply connected.*

(b) *For any presentation $A \cong kQ_A/I_A$, the fundamental group of any connected full convex bound subquiver of $(Q_A, I_A)$ is trivial.*

(c) *For any presentation $A \cong kQ_A/I_A$, each irreducible cycle in $Q_A$ is an irreducible contour, and each irreducible contour is naturally contractible.*

(d) *There exists a presentation $A \cong kQ_A/I_A$ such that each irreducible cycle in $Q_A$ is an irreducible contour, and each irreducible contour is naturally contractible.*

*Proof.* Since (b) implies (a) and (c) implies (d) trivially, it suffices to show that (a) implies (c), that (c) implies (b), and that (d) implies (a).

Assume that $A$ is strongly simply connected. By Lemma 1.2, $A$ satisfies the separation condition. To show (c), let $A \cong kQ_A/I_A$ be an arbitrary presentation. Assume that there exists an irreducible cycle $w$ in $Q_A$ that is not an irreducible contour. Then $w$ is not a contour, and hence is of the form $w = pp_1^{-1}vq_1q^{-1}$, where $x$ is a source on $w$, and $p : x \to \cdots \to a$, $p_1 \colon c_1 \to a_1 \to \cdots \to a$ are paths, $v : c_1 — c_2 — \cdots — c_{n-1} — c_n$ is a reduced walk with $c_1, c_n$ sources on $w$, and $q_1 : c_n \to b_1 \to \cdots \to b$, $q : x \to \cdots \to$

$b$ are paths:

$$\begin{tikzcd}
& a & a_1 \arrow[l, squiggly] & c_1 \arrow[l] \arrow[d, dash] \\
x \arrow[ur, squiggly] \arrow[ddr, squiggly] & & & c_2 \arrow[d, dotted, dash] \\
& & & c_{n-1} \arrow[d, dash] \\
& b & b_1 \arrow[l, squiggly] & c_n \arrow[l]
\end{tikzcd}$$

Since $w$ is irreducible, there exists no path in $Q_A$ from $x$ to $c_i$ for each $1 \le i \le n$. If there exist nontrivial paths $p_2\colon x \to \cdots \to y$, $p_3 : y \to \cdots \to a$ and $p_4 : y \to \cdots \to b$, then the cycle $p_3 p_1^{-1} v q_1 p_4^{-1}$ also satisfies the condition that there exists no path in $Q_A$ from $y$ to $c_i$ for each $1 \le i \le n$. Since $Q_A$ is locally finite, we may assume without loss of generality that any path in $Q_A$ from $x$ to $a$ does not meet the paths from $x$ to $b$. Let $(Q, I)$ be the full bound subquiver of $(Q_A, I_A)$ generated by the points lying on a path between points of the cycle $w$. Thus, $Q$ is the convex hull of $w$ in $Q_A$. Let $x \to \cdots \to z$ be a nontrivial path in $Q$; then $z$ cannot be a predecessor of the $c_i$. Then $z$ is a predecessor of exactly one of $a, b$, say of $a$. Let $\alpha : z \to z'$ be an arrow in $Q$; then again $z'$ is a predecessor of $a$, and not a predecessor of $b$. It is now clear that $x$ is not a separating object in the full convex subcategory $kQ/I$ of $A$, a contradiction.

Suppose that there exists an irreducible contour $(p, q)$ in $Q_A$ from $x$ to $y$ that is not naturally contractible. We may assume that $(p, q)$ is minimal with this property with respect to the partial order defined in Section 1.5. Let $(Q, I)$ be the full bound subquiver of $(Q_A, I_A)$ generated by the points lying on the paths in $Q_A$ from $x$ to $y$, and let $B = kQ/I$. Then $B$ is a connected full convex subcategory of $A$. Let $P_1$ be the set of nontrivial paths in $Q$ that start with $x$ and are contained in a path naturally homotopic to $p$ in $(Q, I)$, and let $P_2$ be the set of nontrivial paths in $Q$, that start with $x$ and are contained in a path that is not naturally homotopic to $p$ in $(Q, I)$. By the minimality of $(p, q)$, we have $P_1 \cap P_2 = \varnothing$, and each path in $Q$ that is reducible to $p$ in $Q$ is in $P_1$. Let $R(x) = \operatorname{rad} P(x)_B$. Then $R(x) = R_1 + R_2$, where, for each $i = 1, 2$, $R_i$ is the $k$-vector space with a basis consisting of the residue classes modulo $I$ of the paths in $P_i$. Now, by definition, any two paths $p_1 \in P_1, p_2 \in P_2$ are not involved in any minimal relation simultaneously. Thus $R_1 \cap R_2 = 0$. Moreover, for any two paths $p_1 \in P_1, p_2 \in P_2$, we know that $p_2$ is not reducible to $p_1$; thus $p_1, p_2$ do not have a common point other than $x, y$. It follows that if $p' : x \to \cdots \to z$ is a path in the $k$-basis of $R_i$, for some $i = 1, 2$, and $\alpha : z \to z'$ is an arrow in $Q$, then $p'\alpha = 0$ or is in the $k$-basis of $R_i$. Therefore, the $R_i$ are submodules of $R(x)$. Thus $x$ is not a separating object of $B$, a contradiction that completes the proof of (c).

We now show that (d) implies (a). Let $A \cong kQ_A/I_A$ be a presentation satisfying (d). It suffices to show that $\pi_1(Q_A, I_A) = 1$ It easily follows from the hypothesis that any contour $(p, q)$ in $Q_A$ is naturally contractible. Let $w$ be a cycle in $Q_A$, and, as in Section 1.5, let $\sigma(w)$ be the number of sources of $w$. If $\sigma(w) = 1$, then $w$ is a contour, and hence is naturally contractible. Assume $\sigma(w) > 1$. Then $w$ is not irreducible by hypothesis. Therefore $w \sim w_1 w_2$, where $w_1, w_2$ are cycles with $\sigma(w) > \sigma(w_1), \sigma(w) > \sigma(w_2)$. Thus $w$ is naturally contractible by induction. It follows easily that any closed walk in $Q$ is naturally contractible. The same argument shows that (c) implies (b). ∎

While proving the above theorem, we have shown that the equivalent conditions of the theorem are also equivalent to the statement that any connected full convex subcategory of our locally bounded category satisfies the separation condition. In fact, we have the following theorem of Skowroński [14] (4.1) whose proof, made for finite-dimensional algebras, extends easily to the case of locally bounded categories.

THEOREM 1.4. *Let $A$ be a connected triangular locally bounded $k$-category. The following conditions are equivalent*:

(a) *$A$ is strongly simply connected.*

(b) *For any connected full convex subcategory $C$ of $A$, we have $H^1(C) = 0$.*

(c) *Any connected full convex subcategory of $A$ satisfies the separation condition.*

(d) *Any connected full convex subcategory of $A$ satisfies the co-separation condition.*

Here and in the sequel, $H^1(C)$ denotes the first Hochschild cohomology group of $C$ with coefficients in the bimodule ${}_C C_C$ (see [8]).

COROLLARY 1.5. *Let $A$ be a connected triangular locally bounded $k$-category. The following conditions are equivalent*:

(a) *$A$ is strongly simply connected.*

(b) *There exists a presentation $A \cong kQ_A/I_A$ such that the fundamental group of any finite connected full convex bound subquiver of $(Q_A, I_A)$ is trivial.*

(c) *For any presentation $A \cong kQ_A/I_A$, the fundamental group of any finite connected full convex bound subquiver of $(Q_A, I_A)$ is trivial.*

(d) *Any connected full convex subcategory of $A$ with finitely many objects satisfies the separation condition.*

(e) *Any connected full convex subcategory of $A$ with finitely many objects satisfies the co-separation condition.*

(f) *For any connected full convex subcategory $C$ of $A$ with finitely many objects, we have $H^1(C) = 0$.*

*Proof.* It suffices to observe that the conditions stated are of a local nature (for instance, any indecomposable projective module is finite-dimensional). ∎

## 2. SCHURIAN STRONGLY SIMPLY CONNECTED LOCALLY BOUNDED CATEGORIES

A locally bounded $k$-category $A$ is called *Schurian* if $\dim_k A(x, y) \leq 1$ for all $x, y \in A_0$. Schurian strongly simply connected finite-dimensional algebras were studied in [9], where they are called completely separating algebras. Furthermore, it is shown in [2] that if $A$ is a Schurian algebra, all of whose indecomposable projective modules are directing, then the following conditions are equivalent:

(a) $A$ is simply connected.

(b) $A$ is strongly simply connected.

(c) $A$ satisfies the separation condition.

Our aim is to find a criterion allowing us to verify whether a Schurian locally bounded $k$-category is strongly simply connected. We start with the following lemma.

LEMMA 2.1. *Let $A$ be a triangular locally bounded $k$-category that is Schurian and strongly simply connected, and let $A \cong kQ_A/I_A$ be any presentation. Then, for any contour $(p, q)$ in $Q_A$, we have that $p \in I_A$ if and only if $q \in I_A$.*

*Proof.* We note that, since $A$ is Schurian, for any contour $(u, v)$ with $u, v \notin I_A$, there exists a nonzero $\lambda \in k$ such that $u = \lambda v$. Assume that there exists a contour $(p, q)$ in $Q_A$ from $x$ to $y$ such that exactly one of $p$ and $q$ lies in $I_A$. We may assume that $(p, q)$ is minimal with respect to the partial order defined in Section 1.5. Suppose that $p \notin I_A$ and that $q \in I_A$. If $(p, q)$ is not irreducible, then there exist paths $p = p_0, p_1, \ldots, p_{m-1}, p_m = q$ from $x$ to $y$ such that $(p_{i-1}, p_i)$ is an interlaced contour for each $1 \leq i \leq m$. It follows from the minimality of $(p, q)$ that $p_1 \notin I_A$ and, inductively, $q \notin I_A$. This contradiction shows that $(p, q)$ is irreducible. By Theorem 1.3, the contour $(p, q)$ must be naturally contractible, that is, there exist paths $p = p_0, p_1, \ldots, p_{m-1}, p_m = q$ in $Q_A$ from $x$ to $y$ such that for each $0 \leq i < m$, $p_i$ and $p_{i+1}$ contain subpaths $q_i$ and $q_{i+1}$, respectively, which are involved in the same minimal relation in $(Q_A, I_A)$. If $q_1 \neq p_1$, then $(p_0, p_1)$ is an interlaced contour, and hence $p_1 \notin I_A$ by the minimality of $(p, q)$. If $q_1 = p_1$, then $p = p_0$ and $p_1$ are involved in the same minimal relation, and hence $p_1 \notin I_A$. Inductively, $q \notin I_A$. This contradiction completes the proof. ∎

LEMMA 2.2. *Let $A$ be a triangular locally bounded $k$-category that is Schurian and strongly simply connected, and let $A \cong kQ_A/I_A$ be any presentation. Then all irreducible cycles in $Q_A$ are irreducible contours and, for each irreducible contour $(p, q)$ in $Q_A$, we have $p, q \notin I_A$ and $p - \lambda q \in I_A$ for some nonzero $\lambda \in k$.*

*Proof.* By Theorem 1.3, all irreducible cycles in $Q_A$ are irreducible contours, and each irreducible contour is naturally contractible. Let $(p, q)$ be an irreducible contour from $x$ to $y$. Since $A$ is Schurian, there exists $\lambda \in k$ such that $p - \lambda q \in I_A$. Assume that one of $p, q$ lies in $I_A$, and further assume that $(p, q)$ is minimal with this property. Let $p \in I_A$. Since $(p, q)$ is naturally contractible, there exist paths $p = p_0, p_1, \dots, p_{m-1}, p_m = q$ in $Q_A$ from $x$ to $y$ such that, for each $1 \leq i \leq m$, $p_{i-1}$ and $p_i$ contain subpaths $q_{i-1}$ and $q_i$, respectively, which are involved in the same minimal relation. Since $(p, q)$ is irreducible, there exists $0 \leq t < m$ such that $p_t$ is reducible to $p$ while $p_{t+1}$ is not. By the minimality of $(p, q)$, we may assume that $p_t \in I_A$. Since $p_{t+1}$ is not reducible to $p$ in $Q_A$, we see that $p_t, p_{t+1}$ have no common point other than $x, y$. Thus $p_t, p_{t+1}$ are involved in the same minimal relation in $(Q_A, I_A)$, and this is impossible. Consequently, neither of $q, p$ lies in $I_A$. ∎

LEMMA 2.3. *Let $Q$ be a connected locally finite quiver without oriented cycles. Then there exists an ascending chain $Q(n)$, with $n \geq 0$, of finite connected full convex subquivers of $Q$ such that*

(a) *$Q(0)$ consists of exactly one point.*

(b) *For each $n > 0$, if $Q(n-1) \subset Q(n)$, then all except one point $x_n$ of $Q(n)$ belong to $Q(n-1)$, and $x_n$ is either a source or a sink in $Q(n)$.*

(c) *$Q = \bigcup_{n \geq 0} Q(n)$.*

*Proof.* Choose any point $x_0$ in $Q$, and let $Q(0) = \{x_0\}$. Suppose that, for an even integer $n$, we have defined an ascending chain $Q(m)$ with $0 \leq m \leq n$ of finite connected full convex subquivers of $Q$ satisfying (a) and (b). If there exists no arrow $a \to b$ in $Q$ with $b$ in $Q(n)$ and $a$ not in $Q(n)$, then we define $Q(n+1) = Q(n)$. Otherwise, let $x_{n+1} \to b$ be an arrow in $Q$ with $b$ in $Q(n)$ and $x_{n+1}$ not in $Q(n)$. Note that, since $Q(n)$ is finite, there exist only finitely many paths in $Q$ starting at $x_{n+1}$ and ending at a point in $Q(n)$. Therefore, we may assume that there exists no path in $Q$ of length greater than one starting with $x_{n+1}$ and ending at a point in $Q(n)$. Furthermore, we may assume that $x_{n+1}$ is such that its distance to $x_0$ (that is, the least length of all of the reduced walks from $x_{n+1}$ to $x_0$) is minimal. Let $Q(n+1)$ be the full subquiver of $Q$ generated by $Q(n)$ and $x_{n+1}$. By our choice of $x_{n+1}$ and the convexity of $Q(n)$, we conclude that $Q(n+1)$ is convex and has $x_{n+1}$ as a source. We now construct $Q(n+2)$ from $Q(n+1)$ in a dual manner so that either $Q(n+2) = Q(n+1)$ or $Q(n+2)$ is generated by $Q(n+1)$ and an additional point $x_{n+2}$, which is a sink of $Q(n+2)$. By induction, we have defined an ascending chain $Q(n)$, with $n \geq 0$, satisfying (a) and (b). Suppose that there exists a point $x$ in $Q$ but not in $\bigcup_{n \geq 0} Q(n)$. Clearly, we may assume that there exists an edge $x - a$ with $a$ in $Q(m)$ for some $m \geq 0$. Assume first that there is an arrow $x \to a$ in $Q$. It follows from our construction that there will be infinitely many arrows starting with $x$, which is impossible. A similar impossibility arises if there is an arrow $a \to x$ in $Q$. ▮

The main result of this section characterizes the Schurian strongly simply connected locally bounded categories in terms of their presentations. In particular, it asserts that a connected triangular locally bounded category is Schurian and strongly simply connected if and only if there exists a presentation such that all irreducible cycles are commutative contours. This result implies that such a category always has a multiplicative basis [4]. A presentation of a Schurian strongly simply locally bounded category $A$, such as that in Theorem 2.4 (b), will be called in the sequel a *normed presentation* of $A$.

THEOREM 2.4. *Let $A$ be a connected triangular locally bounded k-category. The following conditions are equivalent*:

(a) *$A$ is Schurian and strongly simply connected.*

(b) *There exists a presentation $A \cong kQ_A/I_A$ such that all irreducible cycles in $Q_A$ are irreducible contours, and, for each irreducible contour $(p, q)$, we have $p, q \notin I_A$ and $p - q \in I_A$.*

(c) *For any presentation $A \cong kQ_A/I_A$, all irreducible cycles in $Q_A$ are irreducible contours, and, for each irreducible contour $(p, q)$, we have $p, q \notin I_A$ and $p - \lambda q \in I_A$ for some nonzero $\lambda \in k$.*

*Proof.* It follows from Lemma 2.2 that (a) implies (c). Since (b) implies (a), by Theorem 1.3 and the definition of Schurian, it suffices to prove that (c) implies (b), that is, to construct a normed presentation of $A$. Given two points $x, y$ in $Q_A$, there is at most one arrow $\alpha : x \to y$ with which we must associate an element $\varphi(\alpha) \in \operatorname{rad} A(x, y) \setminus \operatorname{rad}^2 A(x, y)$. We say that a path $p$

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \to \cdots \xrightarrow{\alpha_m} x_m$$

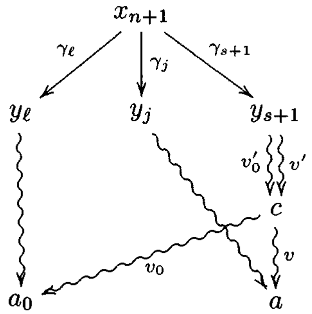
is nonzero if, for each $1 \le i \le m$, there exists $\varphi(\alpha_i) \in \operatorname{rad} A(x_{i-1}x_i) \setminus \operatorname{rad}^2 A(x_{i-1}, x_i)$ such that the composite $\varphi(\alpha_1)\varphi(\alpha_2)\cdots\varphi(\alpha_m)$ (which we write $\varphi(p)$ for the sake of brevity) is nonzero. We call a contour $(p, q)$ nonzero if both $p$ and $q$ are nonzero. Given a contour $(p, q)$ from $x$ to $y$, we say that $(p, q)$ starts with a pair $(\alpha, \beta)$ of arrows if $\alpha, \beta$ are the unique arrows of source $x$ such that $p = \alpha p'$, $q = \beta q'$ with $p', q'$ paths of target $y$.

To construct the required normed presentation, we consider an ascending chain $Q(n)$, with $n \ge 0$, of finite connected full convex subquivers of $Q_A$ satisfying the conditions of Lemma 2.3, and construct $\varphi$ by induction on $n$. Assume thus that for each arrow $\alpha : x \to y$ in $Q(n)$, we have chosen $\varphi(\alpha) \in \operatorname{rad} A(x, y) \setminus \operatorname{rad}^2 A(x, y)$ such that, for any nonzero contour $(p, q)$ in $Q(n)$, we have $\varphi(p) = \varphi(q)$. Assume that $Q(n+1) \ne Q(n)$. It suffices, by duality, to consider the case where $x_{n+1}$ is a source of $Q(n+1)$. Let $\gamma_i$: $x_{n+1} \to y_i$, with $1 \le i \le t$, be all the arrows in $Q(n+1)$ having $x_{n+1}$ as a source. We may clearly assume the $\gamma_i$ to be ordered so that, for each $1 \le i \le t$, if there is no nonzero contour in $Q(n+1)$ starting with $(\gamma_i, \gamma_{j_0})$, then there is no nonzero contour starting with $(\gamma_i, \gamma_j)$ for any $j_0 < j \le t$. We define $\varphi(\gamma_i)$ by induction on $i$, where $1 \le i \le t$.

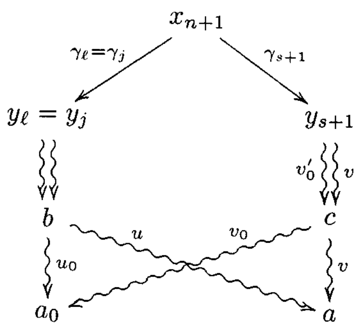
We choose arbitrarily $\varphi(\gamma_1) \in \operatorname{rad} A(x_{n+1}, y_1) \setminus \operatorname{rad}^2 A(x_{n+1}, y_1)$. Let $(p, q)$ be a nonzero contour in $Q(n+1)$ starting with $(\gamma_1, \gamma_1)$; that is, $p = \gamma_1 p'$, $q = \gamma_1 q'$ with $(p', q')$ a nonzero contour in $Q(n)$. Thus $\varphi(p') = \varphi(q')$ and hence $\varphi(p) = \varphi(q)$. Assume that $1 \le s \le t$ and that, for each $1 \le i \le s$, we have chosen $\varphi(\gamma_i) \in \operatorname{rad} A(x_{n+1}, y_i) \setminus \operatorname{rad}^2 A(x_{n+1}, y_i)$ such that, for any nonzero contour in $Q(n+1)$ starting with $(\gamma_i, \gamma_j)$ with $1 \le i,\ j \le s$, we have $\varphi(p) = \varphi(q)$. We wish to define $\varphi(\gamma_{s+1})$. If, for any $1 \le i \le s$, there is no nonzero contour in $Q(n+1)$ starting with $(\gamma_i, \gamma_{s+1})$, then we choose arbitrarily $\varphi(\gamma_{s+1}) \in \operatorname{rad} A(x_{n+1}, y_{s+1}) \setminus \operatorname{rad}^2 A(x_{n+1}, y_{s+1})$. Otherwise, let $(p_0, q_0)$ be a nonzero contour from $x_{n+1}$ to $a_0$ starting with $(\gamma_l, \gamma_{s+1})$ for some $1 \le l \le s$, which we can assume to be minimal with this property. We choose $\varphi(\gamma_{s+1})$ so that $\varphi(p_0) = \varphi(q_0)$. We claim that there exists no nonzero contour $(p, q)$ starting with $(\gamma_j, \gamma_{s+1})$, where $1 \le j \le s+1$, such that $\varphi(p) = \lambda\varphi(q)$ for some $\lambda \ne 1$.

Indeed, assume, on the contrary, that such a contour $(p, q)$ exists with source $x_{n+1}$ and target $a$ (say). We may also assume that $(p, q)$ is minimal with this property. By the induction hypothesis, we have that $j \le s$, and that $a$ is neither a predecessor nor a successor of $a_0$ in $Q_A$, since otherwise $\varphi(p_0) = \varphi(q_0)$ implies $\varphi(p) = \varphi(q)$. Thus, there exists a point $c$ on both $q_0$ and $q$ distinct from $x_{n+1}, a_0, a$ such that the subpath $v_0$ of $q_0$ from $c$ to $a_0$, and the subpath $v$ of $q$ from $c$ to $a$ have no common point except $c$. Let $v', v'_0$ be the subpaths of $q, q_0$ from $y_{s+1}$ to $c$ such that

$q = \gamma_{s+1} v' v$ and $q_0 = \gamma_{s+1} v'_0 v_0$.



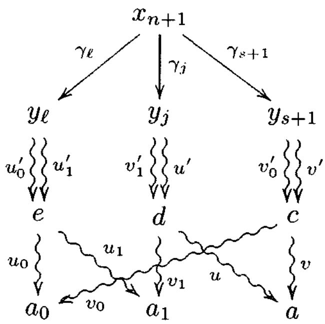

Suppose that $j = l$. Then there exists a point $b$ on both $p_0$ and $p$ other than $x_{n+1}, a_0, a$ such that the subpath $u_0$ of $p_0$ from $b$ to $a_0$ and the subpath $u$ of $p$ from $b$ to $a$ have no common points except $b$. By the minimality of $(p_0, q_0)$, there is no path from $b$ to $c$ and no path from $c$ to $b$ in $Q_A$. Therefore $uv^{-1}v_0u_0^{-1}$ is an irreducible cycle in $Q_A$ that is not a contour, a contradiction of our hypothesis:




Suppose now that $j \neq l$. By our hypothesis on the enumeration of the $\gamma_i$, there exists at least one nonzero contour starting with $(\gamma_l, \gamma_j)$. Thus there exist points $d$ on $p$ and $e$ on $p_0$ such that there exists a nonzero contour $(p_1, q_1)$ from $x_{n+1}$ to $a_1$ (say) starting with $(\gamma_l, \gamma_j)$ and containing $d, e$, and any pair of points $(x, y) \neq (d, e)$ such that $x$ is on the subpath of $p$ from $d$

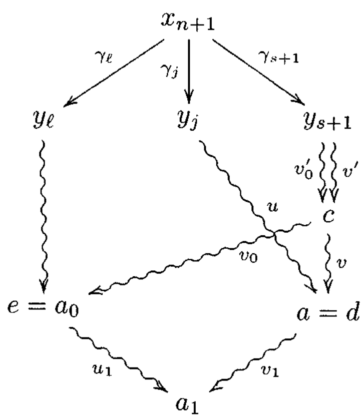
to $a$, and $y$ is on the subpath of $p_0$ from $e$ to $a_0$, does not enjoy this property.

Write $p_1 = \gamma_l u_1' u_1$, $q_1 = \gamma_j v_1' v_1$, $p = \gamma_j u' u$, $q = \gamma_{s+1} v' v$, $p_0 = \gamma_l u_0' u_0$, $q_0 = \gamma_{s+1} v_0' v_0$, where $v_1, u$ have source $d$; $u_0, u_1$ have source $e$; and $v, v_0$ have source $c$:

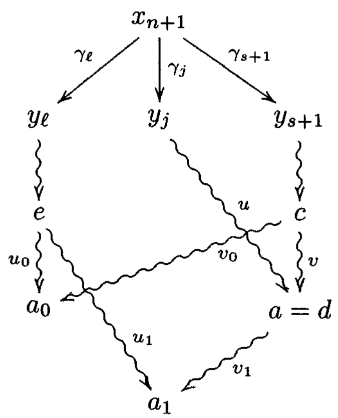



We then have four cases and show that each leads to a contradiction.

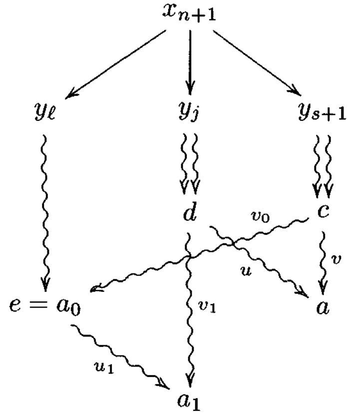
(i) Assume $d = a$ and $e = a_0$. note that $\varphi(v') = \varphi(v_0')$, since both paths lie in $Q(n)$. Then $\varphi(pv_1) = \varphi(p_0 u_1) = \varphi(q_0 u_1) = \varphi(\gamma_{s+1} v_0' v_0 u_1) = \varphi(\gamma_{s+1} v' v v_1) = \varphi(q)\varphi(v_1) = \lambda\varphi(p)\varphi(v_1) = \lambda\varphi(pv_1)$. By Lemma 2.1, we have $\varphi(pv_1) \neq 0$ because $\varphi(p_1) \neq 0$. Therefore $\lambda = 1$, a contradiction.



(ii) Assume $d = a$ and $e \neq a_0$. There is no path from $a_0$ to $a_1$, by the choice of $e$, and no path from $a_1$ to $a_0$, because $a_0$ is not a successor of $a$. Moreover, there are no paths from $c$ to $e$ or from $e$ to $c$ by the minimality of $(p_0, q_0)$. Therefore $vv_1u_1^{-1}u_0v_0^{-1}$ is an irreducible cycle in $Q_A$ that is not a contour, a contradiction of our hypothesis:




(iii) Assume $d \neq a$ and $e = a_0$. There is no path from $a$ to $a_1$, by the choice of $d$, and no path from $a_1$ to $a$, because $a$ is not a successor of $a_0$. Moreover, there are no paths from $e$ to $c$ or from $c$ to $d$, since otherwise $\varphi(p) = \varphi(q)$ by the minimality of $(p, q)$ and the induction hypothesis. Therefore $vu^{-1}v_1u_1^{-1}v_0^{-1}$ is an irreducible cycle in $Q_A$ that is not a contour, a contradiction of our hypothesis:



(iv) Assume $d \neq a$ and $e \neq a_0$. There are no paths from $d$ to $a_0$ or from $e$ to $a$, by the choice of $d, e$. If there exists a path $w$ from $c$ to $a_1$, then $uv^{-1}wv_1^{-1}$ is an irreducible cycle that is not a contour (because the minimality of $(p, q)$ implies that there is no path from $d$ to $c$ or from $c$ to $d$, and the choice of $d, e$ implies that there is no path from $a_1$ to $a$, or from $a$ to $a_1$), and this is a contradiction. Thus, there is no path from $c$ to $a_1$, and $uv^{-1}v_0u_0^{-1}u_1v_1^{-1}$ is an irreducible cycle that is not a contour, again a contradiction.

The theorem is now established by induction.

Let $A$ be a finite-dimensional $k$-algebra that is representation-finite. It is well known (and easy to see) that if $A$ is triangular, then $A$ is Schurian. Hence, for any presentation $A \cong kQ_A/I_A$ and any contour $(p, q)$ in $Q_A$ with $p, q \notin I_A$, there exists a nonzero $\lambda \in k$ such that $p - \lambda q \in I_A$. On the other hand, $A$ is simply connected if and only if it is strongly simply connected (see [7]). Hence Theorems 1.3 and 2.4 yield immediately the following new characterization, in terms of their bound quivers, of simply connected representation-finite algebras.

COROLLARY 2.5. *Let $A$ be a connected finite-dimensional $k$-algebra that is representation-finite. The following conditions are equivalent*:

(a) *$A$ is simply connected.*

(b) *$A$ is triangular, and there exists a presentation $A \cong kQ_A/I_A$ such that all irreducible cycles in $Q_A$ are irreducible contours and, for each irreducible contour $(p, q)$, we have $p, q \notin I_A$ and $p - \lambda q \in I_A$ for some nonzero $\lambda \in k$.*

(c) *$A$ is triangular, and, for any presentation $A \cong kQ_A/I_A$, all irreducible cycles in $Q_A$ are irreducible contours, and, for each irreducible contour $(p, q)$, we have $p, q \notin I_A$ and $p - \lambda q \in I_A$ for some nonzero $\lambda \in k$.*

## 3. CONSTRUCTION OF STRONGLY SIMPLY CONNECTED ALGEBRAS

We recall that the one-point extension of a finite-dimensional algebra $B$ by a $B$-module $M$ is the matrix algebra,

$$A = B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix},$$

where the operations are induced from the matrix operations and the module structure of $M$. The quiver $Q_A$ of $A$ then contains the quiver $Q_B$ of $B$ as a full convex subquiver, and there is an additional (extension) point that is a source. Dually, one defines the one-point coextension $[M]B$ of $B$ by $M$.

Let $A$ be a strongly simply connected algebra. Since $A$ is triangular, it can be constructed by repeated one-point extensions or coextensions. If $A = B[M]$ is strongly simply connected, then $B$ is a full convex subcategory of $A$, and hence is itself strongly simply connected. We are interested in finding necessary and sufficient conditions on a module $M$ over a strongly simply connected algebra $B$ so that the one-point extension $B[M]$, or the one-point coextension $[M]B$, is also strongly simply connected. This would give an inductive construction of strongly simply connected algebras. We start, however, by answering a more general question.

THEOREM 3.1. *Let $B$ be an algebra (not necessarily connected), and $M$ be a $B$-module. Then*

(a) $A = B[M]$ *satisfies the separation condition if and only if $B$ satisfies the separation condition and $M$ is a separated $B$-module.*

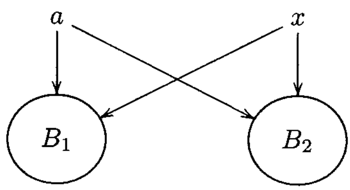
(b) $A = [M]B$ *satisfies the co-separation condition if and only if $B$ satisfies the coseparation condition and $M$ is a separated $B$-module.*

*Proof.* We only prove (a), since the proof of (b) is similar. Assume first that $A$ satisfies the separation condition. Since the extension point $a$ is separating as an object in $A$, the $B$-module $M$ is clearly separated. To prove that $B$ satisfies the separation condition, we must show that any $x \in B_0$ is separating. As usual, we denote by $A^x$ (or $B^x$) the full subcategory of $A$ (or $B$) generated by the nonpredecessors of $x$ in $A$ (or $B$, respectively). Since there is no path from $x$ to $a$, the indecomposable projective $B$-module $P(x)_B$, when considered as an $A$-module, is equal to $P(x)_A$. If $a$ is a predecessor of $x$, then $B^x = A^x$ and $x$ is separating in $B$, because it is so in $A$. If $a$ is not a predecessor of $x$, then $A^x$ is generated by $B^x$ and $a$. Assume rad $P(x)_B$ is not separated. Then there exist two distinct indecomposable summands $R_1, R_2$ of rad $P(x)$ whose supports lie in the same connected component of $B^x$. But $R_1, R_2$ lie in distinct connected components of $A^x$, an impossibility.

Conversely, assume that $B$ satisfies the separation condition and that $M$ is a separated $B$-module. Since the extension point $a$ is clearly a separating object in $A$, we must prove that every $x \in B_0$ such that $x \neq a$ is also separating in $A$. If $a$ is a predecessor of $x$, then clearly $x$ is separating in $A$, since $(A^x)_0 \cup \{x\} = (B^x)_0 \cup \{x\}$ in this case. Thus, we need only consider the case where $a$ is not a predecessor of $x$. In this case, again, $A^x$ is generated by $B^x$ and $a$. Assume, on the contrary, that rad $P(x)_A$ is not a separated $A^x$-module.

Then there exist two distinct indecomposable summands $R_1, R_2$ (say) of rad $P(x)_A$ whose supports are connected in $A^x$. Since they are not connected in $B^x$ (because $B$ satisfies the separation condition), there exist two distinct connected components of $B^x$, say $B_1$ and $B_2$, containing, respectively, the supports of $R_1$ and $R_2$, and connected in $A^x$ through the extension point $a$. In fact, each of $B_1$ and $B_2$ is connected to $a$ by a single arrow: let $a \to a_1 — a_2 — \cdots — a_t$, with $a_t$ in $B_i$, be a walk of least length from $a$ to $b_i$ (where $i = 1, 2$). This hypothesis implies $a_j \neq a$ for all $1 \leq j \leq t$; thus $a_j$ belongs to $B^x$ for all $j$, and hence $a_1$ lie in $B_i$. Therefore the restriction of $M$ to each $B_i$ is nonzero.

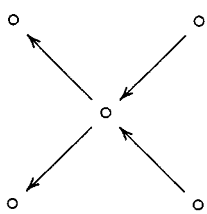
We thus have the following situation:




At this point, it is important to observe that $B_1$ and $B_2$ belong to the same connected component of $B$, since they are connected through $x$. Moreover, the restriction of $M$ to this component is indecomposable, since $M$ is separated as a $B$-module. In particular, there exists a walk $b_1 — c_1 — \cdots — c_r — b_2$ in Supp $M$, with $b_1$ in $B_1, b_2$ in $B_2$, and $c_j$ not in $B^x$ for all $j$ $(1 \leq j \leq r)$ because $B_1, B_2$ are disconnected in $B^x$. Thus, each $c_j$ is a predecessor of $x$. On the other hand, since $c_j$ lies in the support of $M$, there exists a path from $a$ to $c_j$. Hence $a$ is a predecessor of $x$, which is the wanted contradiction. ∎

*Remarks.* (a) Theorem 3.1 generalizes [2 (2.5)].

(b) Let $B$ be an algebra satisfying the separation condition, and $M$ be a separated $B$-module. Then $[M]B$ usually does not satisfy the separation condition (even if $B$ also satisfies the coseparation condition). Indeed, let $B$ be the tame hereditary algebra given by the quiver



and $H$ be a simple homogeneous $B$-module; then $[H]B$ does not satisfy the separation condition.

We have the following easy corollary.

COROLLARY 3.2. *A triangular algebra $A$ satisfies the separation (or the co-separation) condition if and only if there exists a sequence of algebras $A_0, A_1, \dots, A_n = A$ with $A_0$ semisimple, and, for each $0 \le i < n$, a separated $A_i$-module $M_i$ such that $A_{i+1} = A_i[M_i]$ (or $A_{i+1} = [M_i]A_i$, respectively).*

Let $B$ be a triangular algebra, and $M$ be a $B$-module. An enumeration $\{x_1, \dots, x_m\}$ of the points of the support $\operatorname{Supp} M$ of $M$ is called an *admissible ordering of sinks* (or *of sources*) if $j > i$ implies that $x_j$ is not a successor (or predecessor, respectively) of $x_i$. The triangularity of $B$ implies that, for each $B$-module $M$, there exists at least one admissible ordering of sinks and one admissible ordering of sources of the points of $\operatorname{Supp} M$. With each admissible ordering of sinks (or of sources) is associated a filtration of $B$ by a sequence of full convex subcategories. Indeed, let $\{x_1, \dots, x_m\}$ be such an admissible ordering of sinks (or of sources) of the points of $\operatorname{Supp} M$; then we define $B^{(0)} = B$ and, for each $0 < i < m$, we let $B^{(i)}$ be the full subcategory of $B$ generated by the nonsuccessors (or nonpredecessors, respectively) of the points $x_1, \dots, x_i$. Clearly, each $B^{(i)}$ is convex, and we have $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(m-1)}$.

DEFINITION 3.3. Let $B$ be a triangular algebra. A $B$-module $M$ is called *completely coseparated* (or *completely separated*) if, for any admissible ordering of sinks (or of sources, respectively) of the points of $\operatorname{Supp} M$ and, for each $0 \le i < m$, the restriction $M^{(i)} = M|_{B^{(i)}}$ of $M$ to $B^{(i)}$ is separated as a $B^{(i)}$-module.

Thus, any uniserial module (in particular, any simple module) is completely coseparated and completely separated. In general, however, these two classes do not coincide. Example are given later.

Clearly, any completely coseparated (or completely separated) module is separated. In particular, if $B$ is connected, any completely coseparated (or completely separated) $B$-module is indecomposable.

We may now state and prove our main result.

THEOREM 3.4. *Let $B$ be a strongly simply connected algebra, and $M$ be a $B$-module. Then*

(a) *$A = B[M]$ is strongly simply connected if and only if $M$ is a completely coseparated $B$-module.*

(b) *$A = [M]B$ is strongly simply connected if and only if $M$ is a completely separated $B$-module.*

*Proof.* We only prove (a), since the proof of (b) is similar. For the necessity, assume that there exists an admissible ordering of sinks $\{x_1, \ldots, x_m\}$ of the points of $\operatorname{Supp} M$ with associated filtration $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(m-1)}$.

For any $0 \le j < m$, let $M_j = M|_{B^{(j)}}$. Then $B^{(j)}[M_j]$ is a full convex subcategory of $A$, and consequently satisfies the separation condition. By Theorem 3.1, $M_j$ is separated as a $B^{(j)}$-module. This completes the proof of the necessity.

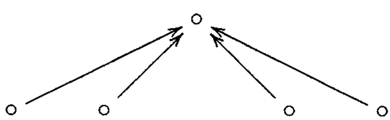
Conversely, we assume that $M$ is completely coseparated and show that each connected full convex subcategory $C$ of $A$ satisfies the separation condition. If the extension point $a$ is not in $C$, then $C$ is a full convex subcategory of $B$, and hence satisfies the separation condition. We thus assume that $a$ lies in $C$.

Let $D$ be the full subcategory of $B$ generated by all objects of $C$ except $a$. Then $C$ is the one-point extension of $D$ by the restriction $M|_D$ of $M$ to $D$. Moreover, $D$ is a full convex subcategory of $B$ and hence satisfies the separation condition. Let $\{x_1, \ldots, x_t\}$, with $t \ge 0$, be the points in $\operatorname{Supp} M$ that do not lie in $C$, and let $\{x_{t+1}, \ldots, x_m\}$ be those lying in $C$. Thus, all of the $x_i$ with $1 \le i \le m$, are successors of the extension point $a$. It then follows from the convexity of $C$ that no point in $C$ is a successor of the $x_i$, with $1 \le i \le t$. In particular, $x_j$ is not a successor of $x_i$ if $1 \le i \le t < j \le m$. We may clearly assume further that $x_j$ is not a successor of $x_i$ whenever $1 \le i < j \le t$ or $t+1 \le i < j \le m$. Therefore $\{x_1, \ldots, x_t, x_{t+1}, \ldots, x_m\}$ is an admissible ordering of sinks of the points of $\operatorname{Supp} M$. Let $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(t)} \supseteq \cdots \supseteq B^{(m-1)}$ be the associated filtration of $B$. Then $M|_{B^{(t)}}$ is separated as a $B^{(t)}$-module. On the other hand, $D$ is a full subcategory of $B^{(t)}$ and $M|_D = M|_{B^{(t)}}$. Thus, $M|_D$ is separated as a $D$-module. It follows from Theorem 3.1 that $C$ satisfies the separation condition. The proof of the theorem is complete. ∎

COROLLARY 3.5. *Let $B$ be a strongly simply connected algebra and $M$ be a completely coseparated* (*or a completely separated*) *$B$-module. Then $M$ is a brick* (*that is*, $\operatorname{End} M \cong k$).

*Proof.* This follows from our theorem and [14 (4.2)]. ∎

EXAMPLE 3.6. (a) Let $B$ be the tame hereditary algebra given by the quiver



Each of the simple homogeneous $B$-modules $H_\lambda$ (with $\lambda \in k \setminus \{0, 1\}$) is completely coseparated, but not completely separated. The algebras $B[H_\lambda]$ (which are just the canonical tubular algebras of type $(2, 2, 2, 2)$) are stronly simply connected. This furnishes an infinite family of strongly simply connected algebras with the same dimension and the same number of isomorphism classes of simple modules.

(b) Let $B$ be given by the quiver

$$\begin{tikzcd}
& 3 \arrow[dl, "\beta"'] & \\
1 & & 5 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
& 4 \arrow[ul, "\delta"] \arrow[dl, "\varepsilon"] & \\
2 & &
\end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$. The $B$-module $M$ given by

$$\begin{tikzcd}
& k \arrow[dl, "1"'] & \\
k & & k \arrow[ul, "1"'] \arrow[dl, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] \\
& k^2 \arrow[ul, "{[10]}"] \arrow[dl, "{[01]}"] & \\
k & &
\end{tikzcd}$$

is indecomposable (and, even, is a brick) but is not completely coseparated. Indeed, if one considers the shown admissible ordering of sinks for the points of $\operatorname{Supp} M = B$, then $B^{(1)}$ is generated by all points except 1 and it is connected, and $M^{(1)} = M|_{B^{(1)}}$ is decomposable, thus not separated. Note that $B$ is strongly simply connected, but $B[M]$ is not.

The following corollary strengthens [2 (5.2)] and [13 (2.2)].

COROLLARY 3.7. *A connected algebra $A$ is strongly simply connected if and only if there exists a sequence of algebras $A_0, A_1, \ldots, A_n = A$, with $A_0 = k$, and for each $0 \leq i < n$, an $A_i$-module $M_i$, such that either $M_i$ is completely coseparated and $A_{i+1} = A_i[M_i]$, or $M_i$ is completely separated and $A_{i+1} = [M_i]A_i$.*

*Proof.* Assume indeed that $A$ is strongly simply connected. By Lemma 2.3, there exists a connected full convex subquiver $Q$ of $Q_A$ such that all of the points of $Q_A$ belong to $Q$ except one, which is a source or a sink. Assume the former, and let $B$ be the connected full convex subcategory of $A$ generated by the points of $Q$; then $A$ is a one-point extension of $B$ by a $B$-module $M$, say. Since $B$ is strongly simply connected (because $A$ is), it follows from our theorem that $M$ is completely coseparated. The proof is completed by induction. ∎

## 4. CONSTRUCTION OF SCHURIAN STRONGLY SIMPLY CONNECTED ALGEBRAS

Since there exists, so far, no general rule to decide whether a given module is indecomposable, we do not have any practical method of deciding whether a given module is completely coseparated, or completely separated, or not. However, if $A$ is a Schurian and strongly simply connected algebra, we can find all $A$-modules $M$ such that $A[M]$, or $[M]A$, is Schurian and strongly simply connected. The result provides an inductive process to construct all Schurian and strongly simply connected algebras with a prescribed number of isomorphism classes of simple modules. In particular, one can obtain in this way all representation-finite simply connected algebras with a prescribed number of isomorphism classes of simple modules.

Let $Q$ be a finite quiver without oriented cycles, and $Q'$ be a full subquiver of $Q$. An enumeration $\{x_1, \ldots, x_m\}$ of the points of $Q'$ is called an *admissible ordering of sinks* (or *of sources*) if $j > i$ implies that $x_j$ is not a successor (or predecessor, respectively). With each such admissible ordering, we associate a filtration of $Q'$ by a sequence of full convex subquivers. Indeed, let $\{x_1, \ldots, x_m\}$ be an admissible ordering of sinks (or of sources) of the points of $Q'$; then we let $Q^{(0)} = Q$, and, for each $0 < i < m$, we let $Q^{(i)}$ be the full subquiver of $Q$ generated by the nonsuccessors (or nonpredecessors, respectively) of $x_1, \ldots, x_i$ in $Q$. Clearly, each $Q^{(i)}$ is convex, and we have $Q^{(0)} \supseteq Q^{(1)} \supseteq \cdots \supseteq Q^{(m-1)}$.

DEFINITION 4.1. Let $Q$ be a finite quiver without oriented cycle. A full subquiver $Q'$ of $Q$ is said to be *completely coseparated* (or *completely separated*) if, for each admissible ordering of sinks (or of sources, respectively) of the points of $Q$, and each $1 \leq i \leq m$, the intersection of $Q'$ with each of the connected components of $Q^{(i)}$ is empty or connected.

LEMMA 4.2. *Let $A$ be a strongly simply connected algebra with ordinary quiver $Q_A$, and $M$ be an $A$-module whose support has quiver $Q$. Then*

(a) *If $M$ is completely coseparated, then $Q$ is a completely coseparated subquiver of $Q_A$.*

(b) *If $M$ is completely separated, then $Q$ is a completely separated subquiver of $Q_A$.*

*Proof.* We only prove (a), since the proof of (b) is similar. Let $\{x_1, \ldots, x_m\}$ be an admissible ordering of sinks of the points of $Q$, and let $A^{(i)}$ denote the full subcategory of $A$ generated by the nonsuccessors of $\{x_1, \ldots, x_i\}$. Then, by definition, $Q_A^{(i)}$ is the quiver of $A^{(i)}$. Since $M$ is completely coseparated, $M^{(i)} = M|_{A^{(i)}}$ is separated as an $A^{(i)}$-module, that is, its restriction to each connected component of $A^{(i)}$ is indecomposable or zero. This implies the statement. ∎

Let $A$ be an algebra with a presentation $A \cong kQ_A/I_A$ and let $Q$ be a full subquiver of $Q_A$. We say that $Q$ is *zero-relation-free* if no path in $Q$ belongs to $I_A$. Given a full subquiver $Q$ of $Q_A$, we denote by $U(Q)$ (see [9] (2.8)) the representation of $Q_A$ defined by

$$U(Q)_x = \begin{cases} k & \text{if } x \in Q_0 \\ 0 & \text{if } x \notin Q_0 \end{cases}$$

$$U(Q)_\alpha = \begin{cases} 1 & \text{if } \alpha \in Q_1 \\ 0 & \text{if } \alpha \notin Q_1. \end{cases}$$

We recall that, by Theorem 2.4, a Schurian strongly simply connected algebra always has a normed presentation.

LEMMA 4.3. *Let $A$ be a Schurian and strongly simply connected algebra, with normed presentation $A \cong kQ_A/I_A$. Let $Q$ be a connected full convex subquiver of $Q_A$ that is zero-relation-free. Then $U(Q)$ has a natural $A$-module structure and is indecomposable.*

*Proof.* To show that $U(Q)$ is an $A$-module, it suffices to show that it is annihilated by the ideal $I_A$. Now, $A \cong kQ_A/I_A$ is a normed presentation; hence all relations are zero-relations or commutativity relations. Since $Q$ is convex and zero-relation-free, the statement follows from the definition of

$U(Q)$. Assume that $U(Q) = M \oplus N$, with $M, N \neq 0$. Since $\dim_k U(Q)_x \leq 1$ for all points $x$ in $Q_A$, every point in $Q$ belongs to Supp $M$ or Supp $N$, and neither support is empty. Assume $x \in (\operatorname{Supp} M)_0$ and $y \in (\operatorname{Supp} N)_0$. Since $Q$ is connected, there is a walk from $x$ to $y$. We may clearly assume, without loss of generality, that there is an edge $x$–$y$, and, even, an arrow $\alpha : x \to y$. But the $U(Q)_\alpha$ must be equal to zero, a contradiction. ∎

We may now state and prove the main result of this section.

THEOREM 4.4. *Let $A$ be a Schurian and strongly simply connected algebra, with normed presentation $A \cong kQ_A/I_A$, and let $M$ be an $A$-module. Then*

(a) *The algebra $A[M]$ is Schurian and strongly simply connected if and only if $M \cong U(Q)$, where $Q$ is a completely coseparated convex subquiver of $Q_A$ that is zero-relation-free.*

(b) *The algebra $[M]A$ is Schurian and strongly simply connected if and only if $M \cong U(Q)$, where $Q$ is a completely separated convex subquiver of $Q_A$ that is zero-relation-free.*

*Proof.* We only prove (a), since the proof of (b) is similar. We first prove the sufficiency. Assume that $A$ satisfies the stated conditions: let $\{x_1, \ldots, x_m\}$ be an admissible ordering of sinks of the points of $Q$, and $A^{(i)}$ be the full subcategory of $A$ generated by the nonsuccessors of $\{x_1, \ldots, x_i\}$. Then $Q_A^{(i)}$ is the quiver of $A^{(i)}$. Let $C$ be a connected component of $A^{(i)}$; then its quiver $Q_C$ is a connected component of $Q_A^{(i)}$. Since $Q$ is a completely coseparated subquiver of $Q_A$, the intersection $Q \cap Q_C$ is empty or is a connected subquiver of $Q_C$, and it is also zero-relation-free. By Lemma 4.3, $U(Q)|_C = U(Q \cap Q_C)$ is an indecomposable $C$-module. This shows that $M$ is a completely coseparated $A$-module and hence, by Theorem 3.4, $A[M]$ is strongly simply connected. Since, clearly, $A[M]$ is Schurian, we are done.

We now prove the necessity. Assume that $B = A[M]$ is Schurian and strongly simply connected, and given a normed presentation $B \cong kQ_B/I_B$ so that $(Q_A, I_A)$ is a full bound subquiver of $(Q_B, I_B)$. We denote by $b$ the extension point of $B$. By Theorem 3.4, the $A$-module $M$ is completely coseparated. Let $Q$ be the quiver of Supp $M$. Since $A$ is Schurian, for any $x \in Q_0$, we have $\dim_k M_x \leq 1$. By Definition 4.1, the quiver $Q$ is completely coseparated. We now prove that $Q$ is convex and zero-relation-free. Let $p : x_0 \to x_1 \to \cdots \to x_t$, with $t \geq 1$, be a path in $Q_A$, with $x_0$ and $x_t$ in $Q$. Then there exist paths $p_1 : b \to \cdots \to x_0$ and $p_2 : b \to \cdots \to x_t$ in $Q_B$ such that $p_1, p_2 \notin I_B$. By Lemma 2.1, we have $p_1 p \notin I_B$. Consequently, $p \notin I_A$. Therefore $Q$ is zero-relation-free. To prove it is convex, we observe that $p_1 p \notin I_B$ implies that, for each $1 \leq i < t$, the composite of $p_1$ with the subpath $x_0 \to \cdots \to x_i$ is not in $I_B$. Hence $x_i \in Q_0$.

Finally, we want to prove that $U(Q)$ and $M$ are isomorphic (for another proof, see [9] (2.9)]). We let $\tilde{Q}$ be the full subquiver of $Q_B$ generated by $b$ and the points of $Q$. Then $\tilde{Q}$ is clearly a connected full convex subquiver of $Q_B$. Moreover, let $p : b \to y_1 \to \cdots \to y_n$ be a path in $\tilde{Q}$. Then $y_n \in Q_0$. Therefore there exists a path $q$ from $b$ to $y_n$ that is not in $I_B$. By Lemma 2.1, $p$ is not in $I_B$ either. This shows that $\tilde{Q}$ is zero-relation-free. By Lemma 4.3, $U(\tilde{Q})$ is a $B$-module. Notice that, since $\tilde{Q}$ is the quiver of $\operatorname{Supp} P(b)$, and $B$ is Schurian, then $\dim_k P(b)_x = 1$ for each $x \in \tilde{Q}_0$. We construct an isomorphism of $B$-modules $\tilde{f} : U(\tilde{Q}) \to P(b)$ in the following way. We define $\tilde{f}_b : U(\tilde{Q})_b \to P(b)_b$ to be the identity on $k = U(\tilde{Q})_b = P(b)_b$. We now let $x \in Q_0$ be arbitrary. There exists a path in $Q_B$

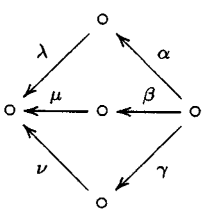
$$b = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_t} x_t = x.$$

For each $1 \le i \le t$, there exists a nonzero scalar $\lambda_{\alpha_i} \in k$ such that $P(b)_{\alpha_i}$ equals the multiplication by $\lambda_{\alpha_i}$. We then define $\tilde{f}_x : U(\tilde{Q})_x \to P(b)_x$ to be the multiplication by $\lambda_{\alpha_1}^{-1} \cdots \lambda_{\alpha_t}^{-1}$. We must show that $\tilde{f}_x$ is well defined. Assume that

$$b = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_s} y_s = x$$

is another path in $Q_B$ from $b$ to $x$. Then $\lambda_{\alpha_1} \cdots \lambda_{\alpha_t} = \lambda_{\beta_1} \cdots \lambda_{\beta_s}$ because $P(b)$ is a $B$-module, and $B$ is given a normed presentation. Hence $\lambda_{\alpha_1}^{-1} \cdots \lambda_{\alpha_t}^{-1} = \lambda_{\beta_1}^{-1} \cdots \lambda_{\beta_s}^{-1}$. Thus $\tilde{f}_x$ is well defined. Clearly $\tilde{f}$ is an isomorphism of $B$-modules that restricts to an isomorphism of $A$-modules $f : U(Q) \to M = \operatorname{rad} P(b)$. ∎

EXAMPLE 4.5. In the non-Schurian case, the support of a completely coseparated module is not necessarily convex. Indeed, let $A$ be given by the quiver



bound by $\alpha\lambda + \beta\mu + \gamma\nu = 0$, and let $M$ be the completely coseparated module given by

$$\begin{tikzcd}
& 0 \arrow[dl] & \\
k & k \arrow[l, "1"'] & k \arrow[ul] \arrow[l, "1"'] \arrow[dl, "1"] \\
& k \arrow[ul, "-1"] &
\end{tikzcd}$$

COROLLARY 4.6. *Let $A$ be a triangular algebra. Then $A$ is Schurian and strongly simply connected if and only if there exists a sequence of algebras $A_0, A_1, \dots, A_n = A$ with $A_0 = k$ and, for each $0 \le i < n$, a full convex zero-relation-free subquiver $Q_i$ of $Q_{A_i}$ such that either $Q_i$ is completely coseparated and $A_{i+1} = A[U(Q_i)]$ or $Q_i$ is completely separated and $A_{i+1} = [U(Q_i)]A_i$.*

*Proof.* This follows from Lemma 2.3 and Theorem 4.4. ∎

COROLLARY 4.7. *For each $n \ge 1$, there exist only finitely many nonisomorphic Schurian strongly simply connected algebras having $n$ isomorphism classes of simple modules.*

*Proof.* This follows from Corollary 4.6 and induction. ∎

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